

Constructive aspects of Riemann's permutation theorem for series

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Abstract

The notions of permutable and weak-permutable convergence of a series $\sum_{n=1}^{\infty} a_n$ of real numbers are introduced. Classically, these two notions are equivalent, and, by Riemann's two main theorems on the convergence of series, a convergent series is permutablely convergent if and only if it is absolutely convergent. Working within Bishop-style constructive mathematics, we prove that Ishihara's principle **BD-N** implies that every permutablely convergent series is absolutely convergent. Since there are models of constructive mathematics in which the Riemann permutation theorem for series holds but **BD-N** does not, the best we can hope for as a partial converse to our first theorem is that the absolute convergence of series with a permutability property classically equivalent to that of Riemann implies **BD-N**. We show that this is the case when the property is weak-permutable convergence.

1 Introduction

This paper follows on from [2], in which the first two authors gave proofs, within the framework of Bishop-style constructive analysis (**BISH**),¹ of the two famous series theorems of Riemann [17]:²

RST₁ *If a series $\sum a_n$ of real numbers is absolutely convergent, then for each permutation σ of the set \mathbf{N}^+ of positive integers, the series $\sum a_{\sigma(n)}$ converges to the same sum as $\sum a_n$.*

RST₂ *If a series $\sum a_n$ of real numbers is conditionally convergent, then for each real number x there exists a permutation σ of \mathbf{N}^+ such that $\sum a_{\sigma(n)}$ converges to x .*

It is not hard to extend the conclusion of **RST₂** to what we call its *full, extended version*, which includes the existence of permutations of the series $\sum a_n$ that diverge to ∞ and to $-\infty$. In consequence, a simple reductio ad absurdum argument

¹That is, analysis using intuitionistic logic, a related set theory such as that of Aczel and Rathjen [1], and dependent choice. For more on **BISH**, see [3, 4, 7].

²We use shorthand like $\sum a_n$ and $\sum a_{\sigma(n)}$ for series when it is clear what the index of summation is.

proves classically that if a real series $\sum a_n$ is **permutably convergent**—that is, every permutation of $\sum a_n$ converges in \mathbf{R} —then it is absolutely convergent. An intuitionistic proof of this last result was provided by Troelstra ([19], pages 95 ff.), using Brouwer’s continuity principle for choice sequences. That result actually has one serious intuitionistic application: Spitters ([18], pages 2101–2) uses it to give an intuitionistic proof of the characterisation of normal linear functionals on the space of bounded operators on a Hilbert space; he also asks whether there is a proof of the Riemann-Troelstra result within **BISH** alone. In Section 3 below, we give a proof, within **BISH** *supplemented* by the constructive-foundationally important principle **BD-N**, that permutable convergence implies absolute convergence. While this proof steps outside unadorned **BISH**, it is valid in both intuitionistic and constructive recursive mathematics, in which **BD-N** is derivable.

This raises the question: over **BISH**, does the absolute convergence of every permutably convergent series imply **BD-N**? Thanks to Diener and Lubarsky [8], we now know that in certain formal systems of **BISH**, the answer is negative; in other words, the result about permutably convergent series is weaker than **BD-N**. In turn, this raises another question: is there a proposition that is *classically* equivalent to, and clearly cognate with, the absolute convergence of permutably convergent series and that, added to **BISH**, implies **BD-N**? In order to answer this question affirmatively, we introduce in Section 2 the notion of *weak-permutable convergence* and then derive some of its fundamental properties, including its classical equivalence to permutable convergence. In Section 4 we show that the absolute convergence of weak-permutably convergent series implies **BD-N**. Thus, in **BISH**, we have the implications

Every weak-permutably convergent series is absolutely convergent

\implies **BD-N**

\implies Every permutably convergent series is absolutely convergent.

In view of the Diener-Lubarsky results in [8], neither of these implications can be reversed.

2 Weak-permutably convergent series in BISH

By a **bracketing** of a real series $\sum a_n$ we mean a pair comprising

- a strictly increasing mapping $f : \mathbf{N}^+ \rightarrow \mathbf{N}^+$ with $f(1) = 1$, and
- the sequence **b** defined by

$$b_k \equiv \sum_{i=f(k)}^{f(k+1)-1} a_i \quad (k \geq 1).$$

We also refer, loosely, to the series $\sum b_k$ as a bracketing of $\sum a_n$.

We say that $\sum a_n$ is **weak-permutably convergent** if it is convergent and if for each permutation σ of \mathbf{N}^+ , there exists a convergent bracketing of $\sum a_{\sigma(n)}$. Clearly, permutable convergence implies weak-permutable convergence. As we shall see in this section, the converse holds classically; later we shall show that it does not hold constructively. As a first step towards this, we have:

Proposition 1. *Let $\sum a_n$ be a weak-permutably convergent series of real numbers, with sum s , and let σ be a permutation of \mathbf{N}^+ . Then every convergent bracketing of $\sum a_{\sigma(n)}$ converges to s .*

The proof of this proposition will depend on some lemmas.

Lemma 2. *Let $\sum a_n$ be a convergent series of real numbers, with sum s , and let σ be a permutation of \mathbf{N}^+ . If there exists a bracketing (f, \mathbf{b}) of $\sum a_{\sigma(n)}$ that converges to a sum $t \neq s$, then there exist a permutation τ of \mathbf{N}^+ and a strictly increasing sequence $(k_i)_{i \geq 1}$ of positive integers such*

$$\left| \sum_{n=f(k_i)+1}^{f(k_{i+1})} a_{\tau(n)} \right| > \frac{1}{3} |s - t| \quad (1)$$

for all i .

Proof. Consider, to illustrate, the case where $s < t$. For convenience, let $\varepsilon \equiv \frac{1}{3}(t - s)$. Pick k_1 such that

$$\left| \sum_{n=j}^k a_n \right| < \frac{\varepsilon}{2} \quad (k > j > f(k_1)).$$

Then $\sum_{n=1}^{f(k_1)} a_n < s + \varepsilon$. Set $\tau(k) \equiv k$ for $1 \leq k \leq f(k_1)$. Next pick $k_2 > k_1$ such that

- $\{1, \dots, f(k_1)\} \subset \{\sigma(n) : 1 \leq n \leq f(k_2)\}$ and
- $\left| \sum_{n=f(j)}^{f(k)} a_{\sigma(n)} \right| < \varepsilon/2$ whenever $k > j > f(k_2)$.

Define $\tau(n)$ for $f(k_1) < n \leq f(k_2)$ so that

$$\{\sigma(n) \mid 1 \leq n \leq f(k_2), \sigma(n) > f(k_1)\} = \{\tau(f(k_1) + 1), \dots, \tau(f(k_2))\}$$

Note that

$$\sum_{n=1}^{f(k_2)} a_{\tau(n)} = \sum_{n=1}^{f(k_2)} a_{\sigma(n)} > t - \varepsilon.$$

Next, pick $k_3 > k_2$ such that

$$\{\tau(1), \dots, \tau(f(k_2))\} \subset \{1, \dots, f(k_3)\}$$

Define $\tau(n)$ for $f(k_2) < n \leq f(k_3)$ so that

$$\{n : 1 \leq n \leq f(k_3), n > \tau(f(k_2))\} = \{\tau(f(k_2) + 1), \dots, \tau(f(k_3))\}.$$

Then

$$\sum_{n=1}^{f(k_3)} a_{\tau(n)} = \sum_{n=1}^{f(k_3)} a_n < s + \varepsilon.$$

Carrying on in this way, we construct, inductively, a strictly increasing sequence $(k_i)_{i \geq 1}$ of positive integers, and a permutation τ of \mathbf{N}^+ , such that for each j ,

$$\sum_{n=1}^{f(k_{2j-1})} a_{\tau(n)} < s + \varepsilon \quad \text{and} \quad \sum_{n=1}^{f(k_{2j})} a_{\tau(n)} > t - \varepsilon.$$

When $i \in \mathbf{N}^+$ is even, we obtain

$$\left| \sum_{n=f(k_i)+1}^{f(k_{i+1})} a_{\tau(n)} \right| \geq \sum_{n=1}^{f(k_i)} a_{\tau(n)} - \sum_{n=1}^{f(k_{i+1})} a_{\tau(n)} > t - s - 2\varepsilon > \frac{1}{3}(t - s).$$

A similar argument gives (1) when i is odd. \square

Lemma 3. *Under the hypotheses of Lemma 2, the series $\sum |a_n|$ diverges.*

Proof. Construct the permutation τ and the sequence $(k_i)_{i \geq 1}$ as in Lemma 2. Given $C > 0$, compute j such that $(j-1)|s-t| > 3C$. Then

$$\sum_{n=1}^{f(k_j)} |a_{\tau(n)}| \geq \sum_{i=1}^{j-1} \left| \sum_{n=f(k_i)+1}^{f(k_{i+1})} a_{\tau(n)} \right| > \frac{j-1}{3} |s-t| > C.$$

Then compute M such that

$$\{a_{\tau(1)}, \dots, a_{\tau(f(k_j))}\} \subset \{a_1, \dots, a_M\}.$$

Then

$$\sum_{n=1}^M |a_n| \geq \sum_{n=1}^{f(k_j)} |a_{\tau(n)}| > C.$$

Since $C > 0$ is arbitrary, the conclusion follows. \square

Lemma 4. *Let $\sum a_n$ be a convergent series of real numbers, and τ a permutation of \mathbf{N}^+ such that $\sum a_{\tau(n)}$ diverges to infinity. Then it is impossible that $\sum a_{\tau(n)}$ have a convergent bracketing.*

Proof. Suppose there exists a bracketing (f, \mathbf{b}) of $\sum a_{\tau(n)}$ that converges to a sum s . Compute $N > 1$ such that

$$\sum_{n=1}^{\nu} a_{\tau(n)} > s + 1 \quad (\nu \geq N). \quad (2)$$

There exists $N_1 > N$ such that

$$\left| \sum_{i=1}^{N_1} \sum_{n=f(i)}^{f(i+1)-1} a_{\tau(n)} - s \right| < 1$$

and therefore

$$\left| \sum_{n=1}^{f(N_1+1)-1} a_{\tau(n)} \right| < s + 1.$$

Since $f(N_1 + 1) > N$, this contradicts (2). \square

Lemma 5. *Let $\sum a_n$ be a weak-permutably convergent series of real numbers, and σ a permutation of \mathbf{N}^+ . Then it is impossible that $\sum |a_{\sigma(n)}|$ diverge.*

Proof. Suppose that $\sum |a_{\sigma(n)}|$ does diverge. Then, by the full, extended version of **RST**₂, there is a permutation τ of \mathbf{N}^+ such that $\sum a_{\tau(n)}$ diverges to infinity. Since $\sum a_n$ is weak-permutably convergent, there exists a bracketing of $\sum a_{\tau(n)}$ that converges. This is impossible, in view of Lemma 4. \square

Arguing with classical logic, we see that if $\sum a_n$ is weak-permutably convergent, then, by Lemma 5, $\sum |a_n|$ must converge; whence $\sum a_n$ is permutably convergent, by **RST**₁.

Returning to intuitionistic logic, we have reached the **proof of Proposition 1**:

Proof. Suppose that there exists a bracketing of $\sum a_{\sigma(n)}$ that converges to a sum distinct from s . Then, by Lemma 3, $\sum |a_n|$ diverges. Lemma 5 shows that this is impossible. It follows from the tightness of the inequality on **R** that every convergent bracketing of $\sum a_{\sigma(n)}$ converges to s . \square

Since permutable convergence implies convergence and is a special case of weak-permutable convergence, we also have:

Corollary 6. *Let $\sum a_n$ be a permutably convergent series of real numbers, and let σ be a permutation of \mathbf{N} . Then $\sum a_{\sigma(n)} = \sum a_n$.*

3 BD-N and permutable convergence

A subset S of \mathbf{N}^+ is said to be **pseudobounded** if for each sequence $(s_n)_{n \geq 1}$ in S , there exists N such that $s_n/n < 1$ for all $n \geq N$ —or, equivalently, if $s_n/n \rightarrow 0$ as $n \rightarrow \infty$. Every bounded subset of \mathbf{N}^+ is pseudobounded; the converse holds classically, intuitionistically, and in recursive constructive mathematics, but Lietz [14] and Lubarsky [15] have produced models of **BISH** in which it fails to hold for inhabited, countable, pseudobounded sets. Thus the principle

BD-N *Every inhabited, countable, pseudobounded subset of \mathbf{N}^+ is bounded*³

is independent of **BISH**. It is a serious problem of constructive reverse mathematics [5, 12, 13] to determine which classical theorems are equivalent to **BISH** + **BD-N**. For example, it is known that the full form of Banach's inverse mapping theorem in functional analysis is equivalent, over **BISH**, to **BD-N**; see [11].

This section is devoted to our version of the **Riemann permutability theorem**:

Theorem 7. *In **BISH** + **BD-N**, every permutably convergent series of real numbers is absolutely convergent.*

³**BD-N** was introduced by Ishihara in [10] (see also [16]).

Proof. Let $\sum_{i=1}^{\infty} a_i$ be a permutably convergent series of real numbers. To begin with, assume that each a_i is rational. Write

$$a_n^+ = \max\{a_n, 0\}, \quad a_n^- = \max\{-a_n, 0\}.$$

Given a positive rational number ε , define a binary mapping ϕ on $\mathbf{N}^+ \times \mathbf{N}^+$ such that

$$\begin{aligned} \phi(m, n) = 0 &\Rightarrow m > n \wedge \sum_{i=n+1}^m a_i^+ \geq \varepsilon, \\ \phi(m, n) = 1 &\Rightarrow m \leq n \vee \sum_{i=n+1}^m a_i^+ < \varepsilon. \end{aligned}$$

We may assume that $\phi(2, 1) = 0$. Let

$$S \equiv \{n : \exists_m (\phi(m, n) = 0)\}.$$

Then S is countable and downward closed. In order to prove that S is pseudobounded, let $(s_n)_{n \geq 1}$ be an increasing sequence in S . We may assume that $s_1 = 1$. Define a map $\kappa : S \rightarrow \mathbf{N}^+$ by

$$\kappa(n) \equiv \min \left\{ m : m > n \wedge \sum_{i=n+1}^m a_i^+ \geq \varepsilon \right\}.$$

Setting $\lambda_1 = 0$, we construct inductively a binary sequence $\lambda \equiv (\lambda_n)_{n \geq 1}$ with the following properties:

$$\forall_n ((\lambda_n = 0 \wedge \lambda_{n+1} = 1) \Rightarrow n+1 \in S) \quad (3)$$

$$\forall_n \exists_m (\lambda_n = 1 \Rightarrow \lambda_{n+m} = 0) \quad (4)$$

$$\forall_n ((\lambda_n = 0 \wedge \lambda_{n+1} = 0) \Rightarrow s_{n+1} \leq n+1) \quad (5)$$

Suppose that $\lambda_1, \dots, \lambda_n$ have been defined such that

$$\forall_{k < n} ((\lambda_k = 0 \wedge \lambda_{k+1} = 1) \Rightarrow k+1 \in S). \quad (6)$$

In the case $\lambda_n = 0$, if $s_{n+1} \leq n+1$, we set $\lambda_{n+1} = 0$; and if $s_{n+1} > n+1$, we set $\lambda_{n+1} = 1$, noting that $n+1 \in S$ since S is downward closed. In the case $\lambda_n = 1$, we define

$$n' \equiv \min \{i \leq n : \forall_j (i \leq j \leq n \Rightarrow \lambda_j = 1)\}.$$

Then the hypothesis (6) ensures that $n' \in S$. If $\kappa(n') = n$, then $\sum_{i=n'+1}^n a_i^+ \geq \varepsilon$ and we set $\lambda_{n+1} = 0$; otherwise, we set $\lambda_{n+1} = 1$. This concludes the inductive construction of the sequence λ . Note that in the case $\lambda_n = \lambda_{n+1} = 1$, this construction will eventually give $\lambda_{n+1+m} = 0$ for some m , since

$$\kappa(n') \geq n+1, \quad \sum_{i=n'+1}^{\kappa(n')-1} a_i^+ < \varepsilon, \quad \text{and} \quad \sum_{i=n'+1}^{\kappa(n')} a_i^+ \geq \varepsilon.$$

Hence the sequence λ has all three properties (3)–(5).

For convenience, if $n \leq m$ and the following hold, we call the interval $I = [n, m]$ of \mathbf{N}^+ a *bad interval*:

- if $n > 1$ then $\lambda_{n-1} = 0$,
- $\lambda_{m+1} = 0$, and
- $\lambda_i = 1$ for all $i \in I$.

Define a permutation σ of \mathbf{N}^+ as follows. If $\lambda_n = 0$, then $\sigma(n) \equiv n$. If $[n, m]$ is a bad interval, then the construction of the sequence λ ensures that $\kappa(n) = m$, so $\sum_{i=n+1}^m a_i^+ \geq \varepsilon$. Let σ map an initial segment $[n, n+k-1]$ of $[n, m]$ onto

$$\{i : n \leq i \leq m \wedge a_i^+ > 0\},$$

and map the remaining elements of $[n, m]$ onto

$$\{i : n \leq i \leq m \wedge a_i^+ = 0\}.$$

Note that for all $n \geq 1$,

$$(\lambda_{n-1} = 0 \wedge \lambda_n = 1) \Rightarrow \exists_{j,k} \left(n \leq j < k \wedge \sum_{i=j+1}^k a_{\sigma(i)} \geq \varepsilon \right). \quad (7)$$

Since $\sum_{i=1}^{\infty} a_{\sigma(i)}$ is convergent, there exists J such that $\sum_{i=j+1}^k a_{\sigma(i)} < \varepsilon$ whenever $J \leq j < k$. In view of (4), we can assume that $\lambda_J = 0$. If $n \geq J$ and $\lambda_J = 1$, then there exists ν such that $J \leq \nu < n$, $\lambda_\nu = 0$, and $\lambda_{\nu+1} = 1$; whence there exist j, k such that $J \leq \nu \leq j < k$ and $\sum_{i=j+1}^k a_{\sigma(i)} \geq \varepsilon$, a contradiction. Thus $\lambda_n = 0$ for all $n \geq J$, and therefore, by (5), $s_n \leq n$ for all $n > J$. This concludes the proof that S is pseudobounded.

Applying **BD-N**, we obtain a positive integer N such that $n < N$ for all $n \in S$. If $m > n \geq N$ and $\sum_{i=n+1}^m a_i^+ > \varepsilon$, then $\phi(m, n) \neq 1$, so $\phi(m, n) = 0$ and therefore $n \in S$, a contradiction. Hence $\sum_{i=n+1}^m a_i^+ \leq \varepsilon$ whenever $m > n \geq N$. Likewise, there exists N' such that $\sum_{i=n+1}^m a_i^- \leq \varepsilon$ whenever $m > n \geq N'$. Thus if $m > n \geq \max\{N, N'\}$, then

$$\sum_{i=n+1}^m |a_i| = \sum_{i=n+1}^m a_i^+ + \sum_{i=n+1}^m a_i^- \leq 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that the partial sums of the series $\sum |a_n|$ form a Cauchy sequence, and hence that the series converges.

It remains to remove the restriction that the terms a_i be rational. In the general case, for each i pick b_i such that $a_i + b_i$ is rational and $0 < b_i < 2^{-i}$. Note that the series $\sum_{i=1}^{\infty} b_i$ converges absolutely and so, by **RST**₁, is permutably convergent. Hence $\sum_{i=1}^{\infty} (a_i + b_i)$ is permutably convergent. By the first part of the proof, $\sum_{i=1}^{\infty} |a_i + b_i|$ is convergent, as therefore is $\sum_{i=1}^{\infty} |a_i|$. \square

4 Weak-permutable convergence and **BD-N**

Diener and Lubarsky [8] have recently constructed topological models showing that the absolute convergence of every permutably convergent series in \mathbf{R} neither implies **BD-N** nor is provable within the Aczel-Rathjen set-theoretic formulation of **BISH** [1], and may therefore be of constructive reverse-mathematical significance

in its own right. Their models lead us to ask: is there a variant of the Riemann permutability theorem that is *classically equivalent* to the original form and that implies **BD-N**? Since weak-permutable and permutable convergence are classically equivalent, the main result of this section provides an affirmative answer:

Theorem 8. *The statement*

(*) *Every weak-permutably convergent series in \mathbf{R} is absolutely convergent*

*implies **BD-N**.*

The hard part of the proof is isolated in the complicated construction in the following lemma.

Lemma 9. *Let $S \equiv \{s_1, s_2, \dots\}$ be an inhabited, countable, pseudobounded subset of \mathbf{N} . Then there exists a sequence $(a_n)_{n \geq 1}$ of nonnegative rational numbers with the following properties.*

- (i) $\sum (-1)^n a_n$ is convergent and weak-permutably convergent.
- (ii) If $\sum a_n$ converges, then S is bounded.

Proof. To perform this construction, we first replace each s_n by $\max \{s_k : k \leq n\}$, thereby obtaining $s_1 \leq s_2 \leq \dots$. Now construct a binary sequence $(\lambda_k)_{k \geq 1}$ such that

$$\begin{aligned} \lambda_k = 0 &\Rightarrow s_{2^{k+1}} = s_{2^k}, \\ \lambda_k = 1 &\Rightarrow s_{2^{k+1}} > s_{2^k}. \end{aligned}$$

For $2^k + 1 \leq n + 1 < 2^{k+1}$, set $a_n = \lambda_k / (n + 1)$. Note that if $\lambda_k = 1$, then $\sum_{n=2^k+1}^{2^{k+1}} a_n > \frac{1}{2}$. In order to show that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges in \mathbf{R} , first observe that if $\lambda_k = 1$ and $2^k < m_1 \leq m_2 \leq 2^{k+1}$, then

$$\left| \sum_{n=m_1}^{m_2} (-1)^n a_n \right| = \left| \sum_{n=m_1}^{m_2} \frac{(-1)^n}{n+1} \right| < \frac{1}{2^k}. \quad (8)$$

If j, k, m_1, m_2 are positive integers with $2^k < m_1 \leq 2^{k+1} \leq 2^j < m_2 \leq 2^{j+1}$, then

$$\begin{aligned} & \left| \sum_{n=m_1}^{m_2} (-1)^n a_n \right| \\ & \leq \left| \sum_{n=m_1}^{2^{k+1}} (-1)^n a_n \right| + \sum_{\substack{k < \nu < j, \\ \lambda_\nu = 1}} \left| \sum_{n=2^\nu+1}^{2^{\nu+1}} (-1)^n a_n \right| + \left| \sum_{n=2^j+1}^{m_2} (-1)^n a_n \right| \\ & \leq \frac{1}{2^k} + \sum_{\substack{k < \nu < j, \\ \lambda_\nu = 1}} \frac{1}{2^\nu} + \frac{1}{2^j} \\ & \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}. \end{aligned}$$

Hence the partial sums of $\sum_{n=1}^{\infty} (-1)^n a_n$ form a Cauchy sequence, and so the series converges to a sum $s \in \mathbf{R}$.

Consider any permutation σ of \mathbf{N}^+ . In order to show that $\sum_{n=1}^{\infty} (-1)^{\sigma(n)} a_{\sigma(n)}$ converges, we construct strictly increasing sequences $(j_k)_{k \geq 1}$ and $(n_k)_{k \geq 1}$ of positive integers such that for each k ,

- (a) $2^{j_k} < n_k < 2^{j_{k+1}}$,
- (b) $\{n : n+1 < 2^{j_k}\} \subset \{\sigma(n) : n+1 < n_k\} \subset \{1, 2, \dots, 2^{j_{k+1}} - 1\}$, and
- (c) $\left| \sum_{n=2^{j_k}}^i (-1)^n a_n \right| < 2^{-k+1}$ for all $k \geq 1$ and $i \geq 2^{j_k}$.

Setting $j_1 = 2$, pick $n_1 > 4$ such that

$$\{1, 2\} \subset \{\sigma(n) : n+1 < n_1\}.$$

Then pick $j_2 > j_1$ such that $n_1 < 2^{j_2}$,

$$\{\sigma(n) : n+1 < n_1\} \subset \{n : n+1 < 2^{j_2}\},$$

and $\left| \sum_{n=2^{j_2}}^i (-1)^n a_n \right| < 2^{-1}$ for all $i \geq 2^{j_2}$. Next pick, in turn, $n_2 > 2^{j_2}$ and $j_3 > j_2$ such that

$$\{n : n+1 < 2^{j_2}\} \subset \{\sigma(n) : n+1 < n_2\} \subset \{n : n+1 < 2^{j_3}\}$$

and $\left| \sum_{n=2^{j_3}}^i (-1)^n a_n \right| < 2^{-2}$ for all $i \geq 2^{j_3}$. Carrying on in this way, we complete the construction of our sequences $(j_k)_{k \geq 1}, (n_k)_{k \geq 1}$ with properties (a)–(c).

Now consider the sequence $(s_{2^{j_k+1}})_{k \geq 1}$. Since S is pseudobounded, there exists a positive integer K_1 such that $s_{2^{j_k+1}} < k$ for all $k \geq K_1$. Suppose that for each positive integer $k \leq K_1$, there exists i_k such that $j_k \leq i_k < j_{k+1}$ and $\lambda_{i_k} = 1$. Then

$$s_{2^{i_1}} < s_{2^{i_2}} < \dots < s_{2^{i_{K_1}}} < s_{2^{j_{K_1+1}}},$$

so $K_1 \leq s_{2^{j_{K_1+1}}} < K_1$, a contradiction. Hence there exists $k_1 \leq K_1$ such that for each i with $j_{k_1} \leq i < j_{k_1+1}$, we have $\lambda_i = 0$, and therefore $a_n = 0$ whenever $2^i \leq n+1 < 2^{i+1}$. Thus $a_n = 0$ whenever $2^{j_{k_1}} \leq n+1 < 2^{j_{k_1+1}}$. It follows that

$$\begin{aligned} \{a_n : n+1 < 2^{j_{k_1}}\} &\subset \{a_{\sigma(n)} : n+1 < n_{k_1}\} \\ &\subset \{a_n : n+1 < 2^{j_{k_1+1}}\} \\ &= \{a_n : n+1 < 2^{j_{k_1}}\} \cup \{a_n : 2^{j_{k_1}} \leq n+1 < 2^{j_{k_1+1}}\} \\ &= \{a_n : n+1 < 2^{j_{k_1}}\} \cup \{0\}. \end{aligned}$$

Without loss of generality, we may assume that $a_1 = 0$. Then

$$\{a_n : n+1 < 2^{j_{k_1}}\} = \{a_{\sigma(n)} : n+1 < n_{k_1}\}.$$

Next consider the sequence $(s_{2^{j_{k_1+k+1}}})_{k \geq 1}$. Since S is pseudobounded, there exists a positive integer K_2 such that $s_{2^{j_{k_1+k+1}}} < k$ for all $k \geq K_2$. Suppose that

for each positive integer $k \leq K_2$, there exists i_k such that $j_{k_1+k} \leq i_k < j_{k_1+k+1}$ and $\lambda_{i_k} = 1$. Then

$$s_{2^{i_1}} < s_{2^{i_2}} < \cdots < s_{2^{i_{K_2}}} < s_{2^{j_{k_1}+K_2+1}},$$

so $K_2 \leq s_{2^{j_{k_1}+K_2+1}} < K_2$, which is absurd. Hence there exists $\kappa \leq K_2$ such that for each i with $j_{k_1+\kappa} \leq i < j_{k_1+\kappa+1}$, we have $\lambda_i = 0$, and therefore $a_n = 0$ whenever $2^i \leq n+1 < 2^{i+1}$. Setting $k_2 \equiv k_1 + \kappa$, we have $a_n = 0$ for all n with $2^{j_{k_2}} \leq n+1 < 2^{j_{k_2}+1}$. Hence

$$\begin{aligned} \{a_n : n+1 < 2^{j_{k_2}}\} &\subset \{a_{\sigma(n)} : n+1 < n_{k_2}\} \\ &\subset \{a_n : n+1 < 2^{j_{k_2}+1}\} \\ &= \{a_n : n+1 < 2^{j_{k_2}}\} \cup \{a_n : 2^{j_{k_2}} \leq n+1 < 2^{j_{k_2}+1}\} \\ &= \{a_n : n+1 < 2^{j_{k_2}}\} \cup \{0\}. \end{aligned}$$

Thus, since $a_1 = 0$,

$$\{a_n : n+1 < 2^{j_{k_2}}\} = \{a_{\sigma(n)} : n+1 < n_{k_2}\}.$$

Carrying on in this way, we construct positive integers $k_1 < k_2 < k_3 < \cdots$ such that for each i ,

$$\{a_n : n+1 < 2^{j_{k_i}}\} = \{a_{\sigma(n)} : n+1 < n_{k_i}\}.$$

Since both σ and σ^{-1} are injective, it readily follows that for each i ,

$$\{\sigma(n) : n_{k_i} \leq n+1 < n_{k_{i+1}}\} = \{m : 2^{j_{k_i}} \leq m < 2^{j_{k_{i+1}}}\}$$

and therefore

$$\left| \sum_{n=n_{k_i}}^{n_{k_{i+1}}-1} (-1)^{\sigma(n)} a_{\sigma(n)} \right| = \left| \sum_{m=2^{j_{k_i}}}^{2^{j_{k_{i+1}}}-1} (-1)^m a_m \right| < \frac{1}{2^{k_i}}.$$

We now see that

$$\sum_{i=1}^{\infty} \sum_{n=n_{k_i}}^{n_{k_{i+1}}-1} (-1)^{\sigma(n)} a_{\sigma(n)}$$

converges, by comparison with $\sum_{i=1}^{\infty} 2^{-k_i}$. Thus $\sum_{n=1}^{\infty} a_n$ is weak-permutably convergent.

Finally, suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then there exists N such that $\sum_{n=N+1}^{\infty} a_n < 1/2$. It follows that $\lambda_n = 0$, and therefore that $s_n = s_{2^N}$, for all $n \geq N$; whence $s_n \leq s_{2^N}$ for all n , and therefore S is a bounded set. \square

The proof of Theorem 8 is now straightforward:

Proof. Given an inhabited, countable, pseudobounded subset S of \mathbf{N} , construct a sequence $(a_n)_{n \geq 1}$ of nonnegative rational numbers with properties (i) and (ii) in Lemma 9. Assuming (*), we see that $\sum a_n$ converges; whence, by property (ii), S is a bounded set. \square

5 Concluding remarks

We have shown that, over **BISH**,

- with **BD-N**, every permutably convergent series is absolutely convergent;
- the absolute convergence of every weak-permutably convergent series implies **BD-N**.

It follows from the latter result that if weak-permutable convergence constructively implies, and is therefore equivalent to, permutable convergence, then the absolute convergence of every permutably convergent series implies, and is therefore equivalent to, **BD-N**. Since the topological models in [8] show that this is not the case, we see that, relative to **BISH**, weak-permutable convergence is a strictly weaker notion than permutable convergence. In fact, the Diener-Lubarsky result shows that there is no algorithm which, applied to any inhabited, countable, pseudobounded subset S of \mathbb{N} and the corresponding weak-permutably convergent series $\sum a_n$ constructed in the proof of Lemma 9, proves that that series is permutably convergent. Nevertheless, weak-permutable convergence and permutable convergence are classically equivalent notions; the constructive distinction between them is that the former implies, but is not implied by, **BD-N**, which in turn implies, but is not implied by, the latter.

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